NONNEGATIVITY OF CURVATURE ALONG GENERALIZED RICCI FLOW

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Abstract. In this note, we construct a series of examples to show that various nonnegative curvature conditions, including Riemannian curvature and Bismut curvature, are not preserved by the generalized Ricci flow.

Contents

1. Introduction

The invariant curvature conditions have been shown to be important to the study of Ricci flows. Hamilton[Ham86] originally proved that the non-negativity of the curvature operator is invariant along Ricci flow. Böhm-Wilking[BW08] further showed that the 2-non-negativity of the curvature operator can also be preserved, which means the sum of any two eigenvalues of the curvature operator is non-negative. Another invariant curvature condition is about isotropic curvature. Positive isotropic curvature (PIC) condition can be preserved along Ricci flow, which is proved by Hamilton[Ham97] in dimension 4, and by Brendle-Schoen[BS09] in all dimensions. Brendle-Schoen[BS09] also showed that PIC1 and PIC2 conditions are also invariant along Ricci flow. For precise definition of these conditions, we refer readers to the Preliminary section. By maximum principle, the infimum of scalar curvature increases along Ricci flow, hence the non-negativity of scalar curvature is also invariant.

As the most significant and natural generalization of the Ricci flow, the generalized Ricci flow's invariant curvature conditions are also important. The generalized Ricci flow, systematically studied by Streets and Tian [Str08, ST13], first appeared in the mathematical literature in [Str08], emerging from research on the renormalization group flow in physics [CFMP85, OSW06]. Formally, a one-parameter family of Riemannian metrics $g(t)$ and a one-parameter family of closed 3-forms $H(t)$ constitute a solution to the generalized Ricci flow on a manifold M^n if they satisfy the following system of equations:

$$\partial_t g = -2\operatorname{Ric} + \frac{1}{2}H^2, \quad \partial_t H = -\Delta_d H.$$

Here, $H^2(X,Y) = g(i_X H, i_Y H)$ denotes a non-negative definite tensor, and $\Delta_d = dd^* + d^*d$ is the Hodge Laplacian induced by the time-dependent metric $g(t)$.

In dimension 3, the non-negativity of the curvature operator is equivalent to that of the sectional curvature, and the 2-non-negativity of the curvature operator is equivalent to that of the Ricci curvature. So the non-negativity of the sectional curvature, Ricci curvature and scalar curvature are both invariant along Ricci flow in dimension 3. For higher dimension, there exists examples showing that Ricci flow does not preserve the non-negativity of sectional curvatures, see [BW07, BK23].

For the generalized Ricci flow, when the dimension is less than or equal to 2, it reduces to the Ricci flow. For dimensions greater than 2, we first show that, unlike the Ricci flow, various Riemannian curvature conditions are be preserved by the generalized Ricci flow.

Theorem 1.1 (cf. Theorem 4.1)**.** *For any $n \geqslant 3$, the following curvature conditions are NOT preserved along generalized Ricci flow:*

(1) *non-negative curvature operator,*

(2) *2-non-negative curvature operator,*
(3) *non-negative sectional curvature, non-negative Ricci curvature, non-negative scalar curvature,*
(4) *weakly PIC, weakly PIC1, weakly PIC2.*

Hence, we need to search for alternative invariant curvature conditions. For the generalized Ricci flow, the curvature of the Bismut connection is the most natural choice, as the flow is evolving along the Bismut-Ricci curvature.

Let (M^n, g) be a Riemannian manifold and H be a closed 3-form. The Bismut connection[Bis89] ∇^B for the triple (M^n, g, H) is the unique connection which is compatible with g and has torsion tensor H. More generally, we will also consider a family of connections that interpolates between the Levi-Civita connection and the Bismut connection, following the approach of Gauduchon[Gau84]. More precisely, we will consider various curvature condition of the connection

$$\nabla^{(s)} = (1-s)\nabla^{LC} + s\nabla^B.$$

Actually, as an extension of Theorem 1.1, we establish

Theorem 1.2 (cf. Theorem 4.2)**.** *For any $n \geqslant 3$ and $s \in \mathbb{R}$, the following curvature conditions for $\nabla^{(s)}$ are NOT preserved along generalized Ricci flow:*

(1) *non-negative curvature operator,*
(2) *2-non-negative curvature operator,*
(3) *non-negative sectional curvature, non-negative Ricci curvature, non-negative scalar curvature,*
(4) *weakly PIC, weakly PIC1, weakly PIC2.*

In general, the lack of a direct relation between the metric g and the three-form H brings obstructions to the existence of invariant curvature conditions for generalized Ricci flow. In complex geometry, under the pluriclosed condition, the generalized Ricci flow coincides with the famous pluriclosed flow introduced by Streets and Tian[ST10], where H represents the torsion of g. Under such condition, it is more possible to find suitable geometrical invariant conditions. It's worth noting that Ustinovskiy[Ust19] has already identified a flow from the same family (Hermitian curvature flow[ST11]) of the pluriclosed flow that preserves the nonnegative/positive Griffith condition.

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2. Preliminary

2.1. **Curvature Conditions.** We first introduce several curvature conditions as follows. From now on, we denote the Levi-Civita connection by ∇ for short.

Definition 2.1 (nonnegative curvature operator)**.** Given (M^n, g, D) a Riemannian manifold and any connection compatible with metric g, the *curvature tensor* $\mathrm{Rm} \in \Gamma(T^*M^{\otimes 4})$ is defined by

$$\mathrm{Rm}^D(X,Y,Z,W) := g\left(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, W\right).$$

The *curvature operator* $\mathcal{R}^D \in \mathrm{Sym}^2(\Lambda^2 TM)$ is defined by

$$\mathcal{R}^D(X\wedge Y, W\wedge Z) := \mathrm{Rm}^D(X,Y,Z,W).$$

It's well defined due to the compatibility with metric g. We say the curvature operator is nonnegative if

$$\mathcal{R}^D(\alpha,\alpha) \geqslant 0, \qquad \text{for any } \alpha \in \Lambda^2 TM.$$

Definition 2.2 (2-nonnegative-curvature operator)**.** The curvature operator is called *2-nonnegative* if the sum of any two eigenvalues of the curvature operator is nonnegative.

Definition 2.3 (sectional, Ricci, scalar curvature)**.** Given (M^n, g, D) a Riemannian manifold with a metric connection, we can define

$$\begin{aligned}
&\sec^D(e_1\wedge e_2) := \mathrm{Rm}^D(e_1,e_2,e_2,e_1), \text{ where } e_1, e_2 \text{ are orthonormal,}\\
&\mathrm{Ric}^D(X,Y) := \mathrm{tr}\,\mathrm{Rm}^D(X,\cdot,\cdot,Y),\\
&R^D := \mathrm{tr}\,\mathrm{Ric}^D(\cdot,\cdot).
\end{aligned}$$

We say the sectional, Ricci, scalar curvature is nonnegative respectively, if

$$\begin{aligned}&\sec^D(e_1\wedge e_2)\geqslant 0,\ \text{for any orthonormal } e_1,e_2,\\ &\operatorname{Ric}^D(X,X)\geqslant 0,\ \text{for any } X\in TM,\\ &R^D\geqslant 0.\end{aligned}$$

Definition 2.4 (PIC, PIC1, PIC2)**.** We say (M^n,g,D) has weakly positive isotropic curvature, weakly PIC for short, if

$$R_{1331}+R_{1441}+R_{2332}+R_{2442}-2R_{1234}\geqslant 0,$$

for any orthonormal 4-frames $\{e_1,e_2,e_3,e_4\}$. Similarly, we say (M^n,g,D) has weakly PIC1, if

$$R_{1331}+\lambda^2R_{1441}+R_{2332}+\lambda^2R_{2442}-2\lambda R_{1234}\geqslant 0,$$

for any $\lambda\in[-1,1]$.

We say (M^n,g,D) has weakly PIC2, if

$$R_{1331}+\lambda^2R_{1441}+\mu^2R_{2332}+\lambda^2\mu^2R_{2442}-2\lambda\mu R_{1234}\geqslant 0,$$

for any $\lambda,\mu\in[-1,1]$.

Next we compute the explicit expressions of Bismut curvature.

Proposition 2.5. *Given (M^n,g,H), then the $\nabla^{(s)}$-curvature tensor is*

$$\begin{aligned}R^{(s)}(X,Y,Z,W)=&R(X,Y,Z,W)+\frac{1}{2}s(\nabla_XH)(Y,Z,W)-\frac{1}{2}s(\nabla_YH)(X,Z,W)\\&+\frac{1}{4}s^2H\left(X,H(Y,Z)^\sharp,W\right)-\frac{1}{4}s^2H\left(Y,H(X,Z)^\sharp,W\right),\end{aligned}$$

hence

$$R^{(s)}(X,Y,Y,X)=R(X,Y,Y,X)-\frac{1}{4}s^2H\left(X,Y,H(X,Y)^\sharp\right).$$

Proof. By direct computation,

$$\begin{aligned}\nabla^{(s)}_X\nabla^{(s)}_YZ=&\nabla^{(s)}_X\left(\nabla_YZ+\frac{1}{2}sH(Y,Z)^\sharp\right)\\=&\nabla_X\left(\nabla_YZ+\frac{1}{2}sH(Y,Z)^\sharp\right)+\frac{1}{2}sH\left(X,\left(\nabla_YZ+\frac{1}{2}sH(Y,Z)^\sharp\right)\right)^\sharp\\=&\nabla_X\nabla_YZ+\frac{1}{2}s\nabla_XH(Y,Z)^\sharp+\frac{1}{2}sH\left(X,\nabla_YZ\right)+\frac{1}{4}s^2H\left(X,H(Y,Z)^\sharp\right)^\sharp.\end{aligned}$$

$$\begin{aligned}R^{(s)}(X,Y,Z)=&\nabla^{(s)}_X\nabla^{(s)}_YZ-\nabla^{(s)}_Y\nabla^{(s)}_XZ-\nabla^{(s)}_{[X,Y]}Z\\=&R(X,Y,Z)+\frac{1}{2}s\nabla_XH(Y,Z)^\sharp+\frac{1}{2}sH\left(X,\nabla_YZ\right)+\frac{1}{4}s^2H\left(X,H(Y,Z)^\sharp\right)^\sharp\\&-\frac{1}{2}s\nabla_YH(X,Z)^\sharp-\frac{1}{2}sH\left(Y,\nabla_XZ\right)-\frac{1}{4}s^2H\left(Y,H(X,Z)^\sharp\right)^\sharp-\frac{1}{2}sH\left([X,Y],Z\right)^\sharp\\=&R(X,Y,Z)+\frac{1}{2}s(\nabla_XH)(Y,Z)^\sharp+\frac{1}{4}s^2H\left(X,H(Y,Z)^\sharp\right)^\sharp\\&-\frac{1}{2}s(\nabla_YH)(X,Z)^\sharp-\frac{1}{4}s^2H\left(Y,H(X,Z)^\sharp\right)^\sharp.\end{aligned}$$

∎

Proposition 2.6. *Given (M^n,g,H), then the Bismut Ricci curvature is*

$$\operatorname{Ric}^{(s)}=\operatorname{Ric}-\frac{1}{4}s^2H^2-\frac{1}{2}sd^*H,$$

hence the symmetric part is

$$\left(\operatorname{Ric}^{(s)}\right)^{\mathrm{sym}}=\operatorname{Ric}-\frac{1}{4}s^2H^2.$$

Proof.

$$\begin{aligned}\operatorname{Ric}^{(s)}(X,Y) =& \operatorname{tr} R^{(s)}(X,\cdot,\cdot,Y) \\ =& \operatorname{Ric}(X,Y) + \frac{1}{2}s(\nabla_X H)(e_i,e_i,Y) - \frac{1}{2}s(\nabla_{e_i} H)(X,e_i,Y) \\ &+ \frac{1}{4}s^2 H\left(X, H(e_i,e_i)^\sharp, Y\right) - \frac{1}{4}s^2 H\left(e_i, H(X,e_i)^\sharp, Y\right),\end{aligned}$$

where

$$\begin{aligned}\frac{1}{2}(\nabla_X H)(e_i,e_i,Y) &= 0, \\ -\frac{1}{2}(\nabla_{e_i} H)(X,e_i,Y) &= \frac{1}{2}(\nabla_{e_i} H)(e_i,X,Y) = -\frac{1}{2}(d^*H)(X,Y), \\ \frac{1}{4}H\left(X, H(e_i,e_i)^\sharp, Y\right) &= 0, \\ -\frac{1}{4}H\left(e_i, H(X,e_i)^\sharp, Y\right) &= -\frac{1}{4}H\left(Y, e_i, H(X,e_i)^\sharp\right) = -\frac{1}{4}H^2(X,Y).\end{aligned}$$

□

2.2. Evolution equation.

Proposition 2.7 ([GFS21], Lemma 5.10)**.** *Suppose $(M^n, g(t), H(t))$ solves the generalized Ricci flow, then*

$$\partial_t \operatorname{Ric}_{jk} = \left(\nabla^2 \operatorname{Ric} + \nabla^2 H^2\right)^i_{ijk} + R^p_{ijk} \operatorname{Ric}^i_p - \operatorname{Ric}^p_j \operatorname{Ric}_{pk} - \tfrac{1}{4} R^p_{ijk}(H^2)^i_p + \tfrac{1}{4}\operatorname{Ric}^p_j (H^2)_{kp},$$

where

$$\begin{aligned}\left(\nabla^2 \operatorname{Ric}\right)^i_{ijk} =& \Delta \operatorname{Ric}_{jk} - R^p_{kij}\operatorname{Ric}^i_p - \operatorname{Ric}^p_k \operatorname{Ric}_{pj}, \\ \left(\nabla^2 H^2\right)^i_{ijk} =& \tfrac{1}{4}\left[-\Delta H^2_{jk} - \nabla_j \nabla_k |H|^2 + \nabla_j (\operatorname{div} H^2)_k + \nabla_k (\operatorname{div} H^2)_j + g^{qi} R^p_{kij}(H^2)_{pq} + g^{qi} R^p_{kiq} H^2_{jp}\right].\end{aligned}$$

Corollary 2.8. *For (M^3, g, H), $H = \phi dV_g$. We have*

$$(\partial_t - \Delta)\operatorname{Ric}_{jk} = -2R^p_{kij}\operatorname{Ric}^i_p - 2\operatorname{Ric}^p_k \operatorname{Ric}_{pj} - \tfrac{1}{2}\left[(\Delta\phi^2)g_{jk} + \nabla_j\nabla_k \phi^2\right].$$

Proof. In this setting, $H^2 = 2\phi^2 g$, $|H|^2 = 6\phi^2$, $\operatorname{div} H^2 = 2\nabla\phi^2$.

$$\begin{aligned}\left(\nabla^2 H^2\right)^i_{ijk} =& \tfrac{1}{4}\Big[-\Delta H^2_{jk} - \nabla_j\nabla_k |H|^2 + \nabla_j(\operatorname{div} H^2)_k + \nabla_k(\operatorname{div} H^2)_j \\ & + g^{qi} R^p_{kij}(H^2)_{pq} + g^{qi} R^p_{kiq} H^2_{jp}\Big] \\ =& \tfrac{1}{4}\left[-2(\Delta\phi^2)g_{jk} - 6\nabla_j\nabla_k\phi^2 + 2\nabla_j\nabla_k\phi^2 + 2\nabla_k\nabla_j\phi^2 + 2R^i_{kij}\phi^2 + 2\operatorname{Ric}_{kj}\phi^2\right] \\ =& \tfrac{1}{2}\left[-(\Delta\phi^2)g_{jk} - \nabla_j\nabla_k\phi^2\right].\end{aligned}$$

$$\begin{aligned}\partial_t \operatorname{Ric}_{jk} =& \left(\nabla^2 \operatorname{Ric} + \nabla^2 H^2\right)^i_{ijk} + R^p_{ijk}\operatorname{Ric}^i_p - \operatorname{Ric}^p_j \operatorname{Ric}_{pk} - \tfrac{1}{2}R^i_{ijk}\phi^2 + \tfrac{1}{2}\operatorname{Ric}_{jk}\phi^2 \\ =& \Delta\operatorname{Ric}_{jk} - R^p_{kij}\operatorname{Ric}^i_p - \operatorname{Ric}^p_k \operatorname{Ric}_{pj} - \tfrac{1}{2}\left[(\Delta\phi^2)g_{jk} + \nabla_j\nabla_k\phi^2\right] + R^p_{ijk}\operatorname{Ric}^i_p - \operatorname{Ric}^p_j\operatorname{Ric}_{pk} \\ =& \Delta\operatorname{Ric}_{jk} - 2R^p_{kij}\operatorname{Ric}^i_p - 2\operatorname{Ric}^p_k\operatorname{Ric}_{pj} - \tfrac{1}{2}\left[(\Delta\phi^2)g_{jk} + \nabla_j\nabla_k\phi^2\right].\end{aligned}$$

□

Proposition 2.9 ([GFS21], Lemma 5.11)**.** *For the generalized Ricci flow,*

$$(\partial_t - \Delta)R = -\frac{1}{2}\Delta|H|^2 + \frac{1}{2}\operatorname{div}\operatorname{div} H^2 + 2\left\langle \operatorname{Ric}, \operatorname{Ric} - \frac{1}{4}H^2\right\rangle.$$

In particular, for (M^3, g, H), $H = \phi dV_g$, we have

$$(\partial_t - \Delta)R = -2\Delta\phi^2 + 2\left\langle \operatorname{Ric}, \operatorname{Ric} - \frac{1}{2}\phi^2 g\right\rangle.$$

Proof. It follows that $|H|^2 = 6\phi^2$ and

$$\frac{1}{2}\operatorname{div}\operatorname{div} H^2 = \operatorname{div}\operatorname{div}(\phi^2 g) = \nabla^2_{ij}(\phi^2)g^{ij} = \Delta\phi^2.$$

□

Proposition 2.10 ([GFS21], Proposition 4.38). *Let M^3 be a three-manifold, and (M^3, g_t, H_t) be a solution of generalized Ricci flow on M. Define $\phi_t \in C^\infty(M)$ satisfying $H_t = \phi_t dV_{g_t}$. Then*

$$\begin{aligned} \partial_t g &= -2\operatorname{Ric} + \phi^2 g, \\ \partial_t \phi &= \Delta\phi + R\phi - \tfrac{3}{2}\phi^3. \end{aligned}$$

Proposition 2.11. *For (M^3, g, H), $H = \phi dV_g$ and $k \in \mathbb{R}$. We have*

$$(\partial_t - \Delta)\left(R - \frac{3}{2}k\phi^2\right) = -2\Delta\phi^2 + 3k|\nabla\phi|^2 + 2\left\langle \operatorname{Ric}, \operatorname{Ric} - \frac{1}{2}\phi^2 g\right\rangle - 3k\phi^2\left(R - \frac{3}{2}\phi^2\right).$$

So

$$\frac{\partial}{\partial t}\left(R - \frac{3}{2}k\phi^2\right) = \Delta\left(R - \frac{3k+4}{2}\phi^2\right) + 3k|\nabla\phi|^2 + 2\left\langle \operatorname{Ric}, \operatorname{Ric} - \frac{1}{2}\phi^2 g\right\rangle - 3k\phi^2\left(R - \frac{3}{2}\phi^2\right).$$

Proof. First by direct computation,

$$(\partial_t - \Delta)\phi^2 = -2|\nabla\phi|^2 + 2R\phi^2 - 3\phi^4.$$

Then it follows from combining Proposition 2.9. □

Remark 2.12. For $k \geqslant 0$, $R - \frac{3}{2}k\phi^2$ is the scalar curvature of $\nabla^{(\sqrt{k})}$. For $k < 0$, it's unknown whether it represents curvature of some connection.

3. Conformal Modification

We firstly construct $\tilde{g}$ by conformally modifying the standard metric on $\mathbb{S}^3$. In dimension 3, the Ricci curvature of $\tilde{g} = e^{2f} g_{S^3}$ is

$$\begin{aligned} \widetilde{\operatorname{Ric}} &= \operatorname{Ric} - \nabla^2 f + df \otimes df - (\Delta f + |df|^2) g \\ &= -\nabla^2 f + df \otimes df + (1 - \Delta f - |df|^2) g. \end{aligned}$$

We first fix a point $P \in \mathbb{S}^3$, and then fix an normal coordinate $(U; x_1, x_2, x_3)$ with respect to P. Moreover, since

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \partial_i f \delta_j^k + \partial_j f \delta_i^k - \partial_l f g^{lk} g_{ij},$$

if $f(P) = 0$, $df(P) = 0$, this coordinate is also the normal coordinate with respect to $\tilde{g}$.

Proposition 3.1. *For any $0 < A < 10^4$, there exists a metric $\tilde{g}$ on $\mathbb{S}^3$ and a fixed point $P \in \mathbb{S}^3$ such that*

1. $\frac{1}{2}\tilde{g} \leqslant \widetilde{\operatorname{Ric}} \leqslant 2\tilde{g}$ *on* $\mathbb{S}^3$, $\widetilde{\operatorname{Ric}}(P) = \tilde{g}(P)$,
2. $\exists r_0 > 0$, *we have* $\widetilde{\operatorname{Ric}} = \tilde{g}$ *outside* $B_{\tilde{g}}(P, 2r_0) \subset U$,
3. *we have* $\left(1 + 23Ar^2 - 20A^2r^6\right)\tilde{g} \leqslant \widetilde{\operatorname{Ric}} \leqslant \left(1 + 33Ar^2\right)\tilde{g}$ *on* $B_0(P, 10^{-1}r_0)$,
4. $\tilde{\Delta}\tilde{R}(P) = 480A$.

Proof. Suppose $B_0(P, r_1) \subset U$, where $B_0 \subset \mathbb{R}^3$ is the Euclidean ball. We take $0 < r_0 < 10^{-1}r_1$ such that $|(g_{ij}) - (\delta_{ij})| < 10^{-2}$, $|\Gamma_{ij}^k| \leqslant 10^{-2}$ in $B_0(P, r_0)$ and $r_0 < 10^{-4}$. Let η be the cut-off function such that

$$\operatorname{supp}(\eta) \subset B_0(P, r_0), \quad \eta|_{B_0(P, \frac{1}{2}r_0)} = 1, \quad |\nabla\eta| \leqslant 10r_0^{-1}, \quad |\nabla^2\eta| \leqslant 10r_0^{-2},$$

then we take $\tilde{g} = e^{2f} g_{\mathbb{S}^3}$, where $f(x) := -\eta(x) \cdot Ar^4$ and $r^2 = d_0^2(x, P) = x_1^2 + x_2^2 + x_3^2$. By direct computation, in $B_0(P, \frac{1}{2}r_0)$,

$$\partial_i f = -4Ar^2 x_i, \; \partial_{ij} f = -8Ax_i x_j - 4Ar^2\delta_{ij}.$$

$$\begin{aligned} -\left(\nabla^2 f\right)_{ij} &= -\left(\partial_{ij} f - \Gamma_{ij}^k \partial_k f\right) = 4A\left(2x_i x_j + r^2\delta_{ij} - 2\Gamma_{ij}^k r^2 x_k\right) \\ &\geqslant Ar^2\left(4 - 10^{-2}\right)\delta_{ij}, \\ -\left(\nabla^2 f\right)_{ij} &\leqslant 4Ar^2\left(3 + 10^{-3}\right)\delta_{ij} \leqslant Ar^2\left(12 + 10^{-2}\right)\delta_{ij}, \end{aligned}$$

since $x_i x_j dx^i dx^j \geqslant 0$. Recall that

$$\widetilde{\operatorname{Ric}} = -\nabla^2 f + df \otimes df + (1 - \Delta f - |df|^2) g,$$

where

$$\begin{aligned}
&-\Delta f = -g^{ij}\left(\partial_{ij} f - \Gamma_{ij}^k \partial_k f\right) \geqslant (20-10^{-2})Ar^2,\\
&-\Delta f \leqslant (20+10^{-2})Ar^2,\\
&|df|^2 \leqslant (1+10^{-2})\delta^{ij}(4Ar^2x_i)(4Ar^2x_j) \leqslant 20A^2r^6.
\end{aligned}$$

So we have

$$\begin{aligned}
\widetilde{\mathrm{Ric}} &\geqslant \left(1+(24-10^{-1})Ar^2 - 20A^2r^6\right) g \geqslant \left(1+23Ar^2-20A^2r^6\right)\tilde{g},\\
\widetilde{\mathrm{Ric}} &\leqslant \left(1+(32+10^{-1})Ar^2\right) g \leqslant \left(1+33Ar^2\right)\tilde{g},
\end{aligned}$$

by $df \otimes df - |df|^2 g \leqslant 0$. Since

$$\tilde{R} = e^{-2f}\left(3 - 4\Delta f - 2|df|^2\right)$$

and $f(P) = |\nabla f|(P) = |\nabla^2 f|(P) = |\nabla^3 f|(P) = 0$, we have

$$\tilde{\Delta}\tilde{R} = -4\Delta\Delta f(P) = 4A\sum_{i,j}\partial_{ii}\partial_{jj} r^4(P) = 480A.$$

□

4. Proof of the Main Theorem

Theorem 4.1. *For any $n \geqslant 3$, the following curvature conditions are not preserved along generalized Ricci flow:*

(1) *non-negative curvature operator,*
(2) *2-non-negative curvature operator,*
(3) *non-negative sectional curvature, non-negative Ricci curvature, non-negative scalar curvature,*
(4) *weakly PIC, weakly PIC1, weakly PIC2.*

Proof. We first consider the case $n = 3$. Proposition 2.9 shows that

$$(\partial_t - \Delta)R = -2\Delta\phi^2 + 2\left\langle \mathrm{Ric}, \mathrm{Ric} - \frac{1}{2}\phi^2 g\right\rangle.$$

We take $(M^3, g_0) = (\mathbb{T}^3, g_{\text{flat}})$ and fix a point $P = [0^3] \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Let $\eta \in C^\infty(\mathbb{T}^3)$ be a cut-off function such that

$$\operatorname{supp}(\eta) \subset B_0(P, \tfrac{1}{2}), \quad \eta|_{B_0(P,10^{-1})} = 1.$$

Then we take $\phi^2(x) = \eta(x)\cdot x_1^2$. Since we have $\mathrm{Rm}(x, 0) \equiv 0$,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} R(P) = -2\Delta\phi^2(P, 0) = -4 < 0.$$

Note that $R(P, 0) = 0$, then we have there exists $\tau > 0$ such that $R(P, \tau) < 0$. For higher dimension, we take $(M^n, g_0) = (\mathbb{T}^n, g_{\text{flat}})$. Assume the natural projection $\pi : \mathbb{T}^n \to \mathbb{T}^3$ maps to the first three factors and $H_0 := \phi dV_{\mathbb{T}^3} \in \Lambda^3(\mathbb{T}^3)$, then take $H := \pi^*(H_0) \in \Lambda^3(\mathbb{T}^n)$. Let $\tilde{P} = [0^n] \in \mathbb{T}^n$.

Since the initial metric is flat, all the listed curvature conditions are satisfied. Note that $R(P, \tau) < 0$ implies that the conditions (1)(2)(3) fail for $g(\tau)$. Since PIC1 and PIC2 are stronger than PIC condition, it suffices to show $g(\tau)$ is not weakly PIC. Assume $\mathrm{Ric}_{33}(P, \tau) < 0$, then we take $\{e_1, e_2, e_3\}$ as orthonormal basis of $T_P\mathbb{T}^3$, and choose $e_4 \in (T_P\mathbb{T}^3)^\perp$. By direct computation,

$$R_{1331} + R_{1441} + R_{2332} + R_{2442} - 2R_{1234} = R_{1331} + R_{2332} = \mathrm{Ric}_{33} < 0.$$

Hence $g(\tau)$ is not weakly PIC. □

Even in dimension three, the non-negative Ricci curvature condition is not equivalent to the non-negative curvature operator condition. Specifically, suppose the eigenvalues of the curvature operator are k_1, k_2, and k_3. In this case, the eigenvalues of the Ricci operator are $\lambda_3 = k_1 + k_2$, $\lambda_2 = k_1 + k_3$, and $\lambda_1 = k_2 + k_3$. By estimating the upper and lower bounds of the Ricci curvature and using the relation $k_i = \frac{1}{2}\sum_{j=1}^3 \lambda_j - \lambda_i$, we will use Proposition 3.1 to construct an example satisfying the nonnegative curvature operator condition.

Theorem 4.2. *For any $n \geqslant 3$ and $s \in \mathbb{R}$, the following curvature conditions for $\nabla^{(s)}$ are NOT preserved along generalized Ricci flow:*

(1) *non-negative curvature operator,*
(2) *2-non-negative curvature operator,*

(3) *non-negative sectional curvature, non-negative Ricci curvature, non-negative scalar curvature,*
(4) *weakly PIC, weakly PIC1, weakly PIC2.*

Proof. Since $\mathrm{Ric}^{(0)} = \mathrm{Ric}$, the case $s = 0$ follows from Theorem 4.2. Now we assume $s \neq 0$. By Proposition 3.1, we have a metric $\tilde{g}$ on $\mathbb{S}^3$ such that

$$\left(1 + 23Ar^2 - 20A^2r^6\right)\tilde{g} \leqslant \widetilde{\mathrm{Ric}} \leqslant \left(1 + 33Ar^2\right)\tilde{g} \text{ on } B_0(P, 10^{-1}r_0),$$

$\widetilde{\mathrm{Ric}}(P) = \tilde{g}(P)$ and $\widetilde{\mathrm{Ric}} = \tilde{g}$ outside $B_{\tilde{g}}(P, 2r_0) \subset U$.

We can take $\phi \in C^\infty(\mathbb{S}^3)$ such that $\frac{1}{2}s^2\phi^2 = 1 + 12Ar^2 - 20A^2r^6$ on $B_0(P, 10^{-1}r_0)$, and $\frac{1}{2}s^2\phi^2 = 0$ outside $B_{\tilde{g}}(P, 2r_0)$. Thus we construct $g(0) = \tilde{g}$, $H(0) = \phi dV_0$ such that

$$0 \leqslant 11Ar^2\tilde{g} \leqslant \left(\mathrm{Ric}^{(s)}\right)^{\mathrm{sym}}_{g(0)} = \widetilde{\mathrm{Ric}} - \frac{1}{2}s^2\phi^2\tilde{g} \leqslant \left(21Ar^2 + 20A^2r^6\right)\tilde{g}.$$

So the $\nabla^{(s)}$-curvature operator satisfies

$$\mathcal{R}^{(s)}_{g(0)} \geqslant \frac{1}{2}\left(2\min \mathrm{Ric}^{(s)}_{g(0)} - \max \mathrm{Ric}^{(s)}_{g(0)}\right) \geqslant \left(Ar^2 - 20A^2r^6\right)\tilde{g} \geqslant \frac{1}{2}Ar^2\tilde{g} \text{ on } B_0(P, 10^{-1}r_0).$$

Note that $\mathcal{R}^{(s)}_{g(0)} \geqslant 1$ outside $B_{\tilde{g}}(P, 2r_0)$, it's easy to modify ϕ such that $\mathcal{R}^{(s)}_{g(0)} \geqslant 0$ holds on the entire $\mathbb{S}^3$.

By Proposition 2.11,

$$\frac{\partial}{\partial t}\left(R - \frac{3}{2}s^2\phi^2\right) = \Delta\left(R - \frac{3s^2+4}{2}\phi^2\right) + 3s^2|\nabla\phi|^2 + 2\left\langle \mathrm{Ric}, \mathrm{Ric} - \frac{1}{2}\phi^2 g\right\rangle - 3s^2\phi^2\left(R - \frac{3}{2}\phi^2\right).$$

Since

$$\Delta\left(\frac{1}{2}s^2\phi^2\right)(P) = \Delta\left(1 + 12Ar^2 - 20A^2r^6\right)(P) = 72A,$$

then

$$\begin{aligned}\left.\frac{\partial}{\partial t}\right|_{t=0}\left(R - \frac{3}{2}s^2\phi^2\right)(P) &= 480A - \frac{3s^2+4}{s^2}\cdot 72A + 6(1 - s^{-2}) - 18\left(1 - s^{-2}\right)\\ &= \frac{24A(11s^2 - 12) + 12(1 - s^2)}{s^2}\end{aligned}$$

If $|s| > 1$, we have $1 - s^2 < 0$, then we can take sufficiently small $0 < A << 1$ such that $\partial_t\left(R - \frac{3}{2}s^2\phi^2\right)(P, 0) < 0$. If $|s| \leqslant 1$, we have $11s^2 - 12 \leqslant -1$, then we can take $A = 10^2$ such that $\partial_t\left(R - \frac{3}{2}s^2\phi^2\right)(P, 0) < 0$.

Note that $\left(R - \frac{3}{2}s^2\phi^2\right)(P, 0) = 0$, so there exists $\tau > 0$, $(R - \frac{3}{2}s^2\phi^2)(P, \tau) < 0$. Then the conclusion follows from the same argument in Theorem 4.1. ∎

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